Week 14 Self-adjoint operator and Spectral theorem

Defn V is a vector space. $T: V \to V$.

$x \in V$ is said to be an *eigenvector* of T if

① $x \neq \vec{0}$

② $T(x) = \lambda x$ for some scalar λ

λ is the associated *eigenvalue*

eg. $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ by

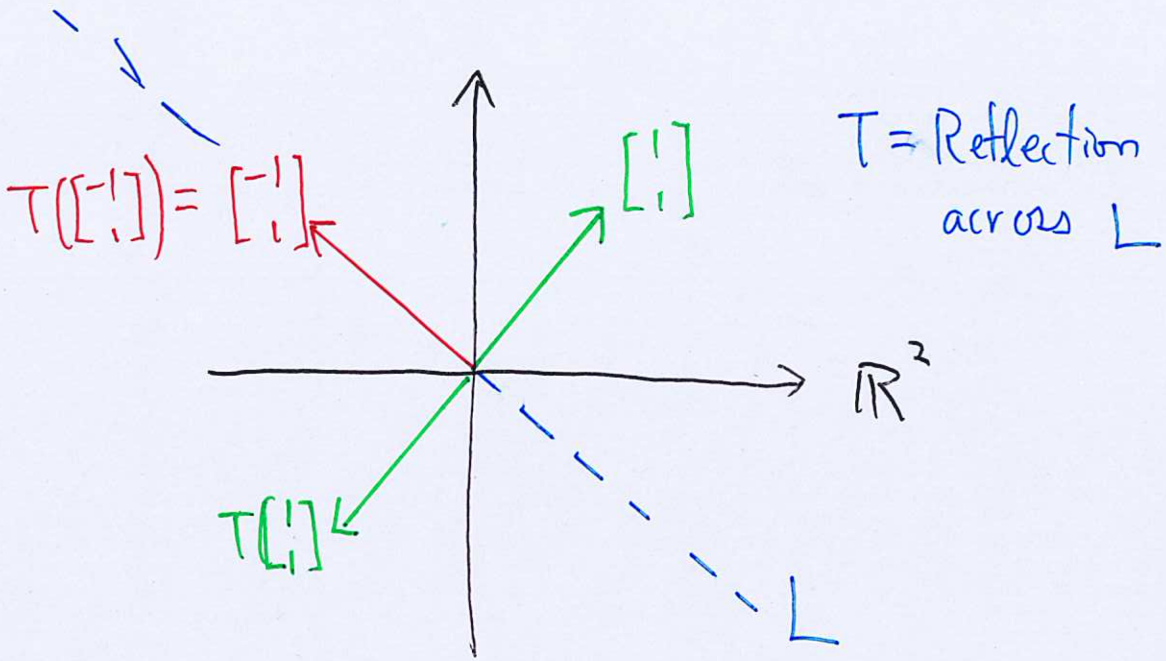
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$$

Note that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is eigenvector of T of associated eigenvalue $-1, 1$ respectively



eg2 $V = \mathcal{C}^{\infty}[-\pi, \pi]$

$f_n(x) = e^{inx}$

$T : V \to V$ defined by

$T(f) = f'$

Then $T(f_n) = f_n' = in\, e^{inx} = in\, f_n$

$\Rightarrow$ f_n is an eigenvector of eigenvalue in

Defn Let H be a Hilbert space, $T : H \to H$ linear

T is said to be

① self-adjoint if $T^* = T$. $\leftarrow$ Our focus

② normal if $T^*T = TT^*$

③ unitary if $T^*T = TT^* = I_H$

Rmk self-adjoint/unitary $\Rightarrow$ normal

eg If $A \in M_{n\times n}(\mathbb{C})$, $T : \mathbb{C}^n \to \mathbb{C}^n$

$T(\vec{x}) = A\vec{x}$ $\quad$ $(T^*(\vec{x}) = A^* x, \; A^* = \bar{A}^t)$

Then T is self adjoint $\Leftrightarrow$ $A^* = A$

eg

$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & 10 \end{bmatrix} \qquad A^* = A$$

Thm If $T: H \to H$ is normal, x, y are eigenvectors of different eigenvalues then $x \perp y$

Pf General case (HW 6)

We prove special case $T = T^*$ here

Suppose $T(x) = \lambda_1 x$

$T(y) = \lambda_2 y$ $\quad \lambda_1 \neq \lambda_2$

$\lambda_1 \langle x, y \rangle = \langle \lambda_1 x, y \rangle = \langle T(x), y \rangle$

$\because T = T^*$ $= \langle x, T(y) \rangle = \langle x, \lambda_2 y \rangle$

$= \overline{\lambda_2} \langle x, y \rangle = \lambda_2 \langle x, y \rangle$

Proved next: $\lambda_2 \in \mathbb{R}$

$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle x, y \rangle = 0 \Rightarrow \langle x, y \rangle = 0$

$\Rightarrow x \perp y$

Thm 3.10-3 Let $T: H \to H$ be bounded

1. If T is self-adjoint, then $\langle T(x), x \rangle \in \mathbb{R}$ $\forall x \in H$
2. If H is complex and $\langle T(x), x \rangle \in \mathbb{R}$ $\forall x \in H$ then T is self-adjoint

Pf 1. $\overline{\langle T(x), x \rangle} = \langle x, T(x) \rangle \overset{T \text{ is self-adjoint}}{=} \langle T(x), x \rangle$

2. Consider $T - T^*$

$\langle (T - T^*)(x), x \rangle = \langle T(x), x \rangle - \langle T^*(x), x \rangle$

$= \langle T(x), x \rangle - \langle x, T(x) \rangle$

$\overset{\langle T(x), x \rangle \text{ is real}}{=} \langle x, T(x) \rangle - \langle x, T(x) \rangle = 0$

HW4 $\Rightarrow T - T^* = 0$

$\Rightarrow T = T^*$

Cor If T is self-adjoint, then all eigenvalues are real

Pf Suppose $T(x) = \lambda x$, $x \neq 0$,

$$① \Rightarrow \langle T(x), x \rangle \in \mathbb{R}$$

$$\cancel{\Rightarrow \lambda \langle \lambda x, x \rangle \in \mathbb{R}}$$

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle T(x), x \rangle$$

$$\Rightarrow \lambda = \frac{\langle T(x), x \rangle}{\langle x, x \rangle} \in \mathbb{R}$$

Thm (Easy version of 9.2-1)

If $T: H \to H$ is a self adjoint operator (bounded)

then $$\|T\| = \sup_{\|x\|=1} |\langle T(x), x \rangle|$$

Pf Let $m = \sup_{\|x\|=1} |\langle T(x), x \rangle|$

Then $\forall x \in H$, $\|x\| = 1$.

$$|\langle T(x), x \rangle| \leq \|T(x)\| \|x\| = \|T(x)\| \leq \|T\|$$

Take supremum among $\|x\| = 1 \Rightarrow m \leq \|T\|$

Next, prove $m \geq \|T\|$

For any $z \neq 0$, then $\frac{z}{\|z\|}$ has length 1

$$\Rightarrow \left| \left\langle T\left(\frac{z}{\|z\|}\right), \frac{z}{\|z\|} \right\rangle \right| \leq m$$

$$\Rightarrow \left| \frac{1}{\|z\|^2} \langle T(z), z \rangle \right| \leq m$$

$$\Rightarrow |\langle T(z), z \rangle| \leq m \|z\|^2$$

This is also true for $z = 0$

$$\Rightarrow |\langle T(z), z \rangle| \leq m \|z\|^2 \quad \forall z \in H \quad \circledast$$

$\forall x, y \in H$

$$\begin{aligned}\langle T(x+y), x+y \rangle &= \langle T(x), x \rangle + \langle T(x), y \rangle \\ &\quad + \langle T(y), x \rangle + \langle T(y), y \rangle \\ &= \langle T(x), x \rangle + \langle T(x), y \rangle \\ &\quad + \langle y, T(x) \rangle + \langle T(y), y \rangle\end{aligned}$$

$T = T^*$

$$\Rightarrow \langle T(x+y), x+y \rangle = \langle T(x), x \rangle + \langle T(y), y \rangle + 2\operatorname{Re}\langle T(x), y \rangle$$

Similarly,

$$\langle T(x-y), x-y \rangle = \langle T(x), x \rangle + \langle T(y), y \rangle - 2\operatorname{Re}\langle T(x), y \rangle$$

Subtraction

$$\Rightarrow 4\operatorname{Re}\langle T(x), y \rangle = \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle$$

$\circledast$

$$\begin{aligned}\Rightarrow 4\operatorname{Re}\langle T(x), y \rangle &\leq m(\|x+y\|^2 + \|x-y\|^2) \\ &= 2m(\|x\|^2 + \|y\|^2)\end{aligned}$$

Replace x by λx, where $\lambda \in \mathbb{C}$ and $|\lambda|=1$

so that $\langle T(x), y\rangle \in \mathbb{R}$ and ≥ 0

$\Rightarrow$ $4\langle T(x), y\rangle \leq 2m(\|x\|^2 + \|y\|^2)$

If $T(x) \neq 0$, put $y = \dfrac{\|x\|}{\|T(x)\|} T(x)$

then $\|y\| = \dfrac{\|x\|}{\|T(x)\|}\|T(x)\| = \|x\|$

$\Rightarrow 4\left\langle T(x), \dfrac{\|x\|}{\|T(x)\|}T(x)\right\rangle \leq 4m\|x\|^2$

$\Rightarrow \dfrac{\|x\|}{\|T(x)\|}\|T(x)\|^2 \leq m\|x\|^2 \quad \begin{pmatrix} x \neq 0 \\ \|x\|>0 \end{pmatrix}$

$\Rightarrow \|T(x)\| \leq m\|x\|$

$\Rightarrow \|T\| \leq m \Rightarrow \boxed{\text{Thm}}$

Spectral theorem for self-adjoint operators

(for $\dim H < \infty$)

Suppose H is a Hilbert space, $\dim H = n < \infty$

$T: H \to H$ is bounded, self-adjoint operator

then $\exists$ orthonormal basis $\{x_1, x_2, \cdots, x_n\}$ of H

such that $T(x_i) = \lambda_i x_i$ where λ_i

are eigenvalues of T and $\lambda_i \in \mathbb{R}$

Matrix version

If $A \in M_{n\times n}(\mathbb{C})$, $A^* = A$, then

$\exists$ unitary matrix Q (i.e. $Q^*Q = I_n$) s.t.

$$Q^*AQ = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Pf By induction on $n = \dim H$

① If $n=1$, trivial

② Assum theorem is true for $n=k$

③ Suppose $\dim H = k+1$,

Note $\|T\| = \sup_{\|x\|=1} |\langle T(x), x\rangle|$

Finite dim $\Rightarrow \exists x \in H, \|x\|=1$ such that

$$|\langle T(x), x\rangle| = \|T\|$$

Also $|\langle T(x), x\rangle| \leq \|T(x)\| \|x\| \leq \|T\|$

Equality holds in Cauchy-Schwarz, $x \neq 0$

$\Rightarrow \exists \lambda$ such that $T(x) = \lambda x$

$\Rightarrow x$ is an eigenvector

Call $x = x_{k+1}$, $\lambda = \lambda_{k+1}$

Consider $H_0 = \operatorname{span}\{x_{k+1}\}^\perp$

then $\dim H_0 = \dim H - 1 = k$

Also, if $x \in H_0$, then

$$\begin{aligned}\langle T(x), x_{k+1}\rangle &= \langle x, T(x_{k+1})\rangle \\ &= \langle x, \lambda_{k+1} x_{k+1}\rangle \\ &= \overline{\lambda_{k+1}} \langle x, x_{k+1}\rangle \\ &= 0 \quad (\because x \in H_0)\end{aligned}$$

$\Rightarrow T(x) \in H_0$

$\Rightarrow T|_{H_0} : H_0 \to H_0$ is an operator on H_0

T is self-adjoint $\Rightarrow T|_{H_0}$ is self-adjoint

Induction assumption

$\Rightarrow \exists$ orthonormal basis $\{X_1, X_2, \cdots, X_k\}$ of H_0

Such that $T(X_i) = \lambda_i X_i$ for $i = 1, 2, \cdots, k$

Then $\{X_1, X_2, \cdots, X_{k+1}\}$ is a basis we want.